Geometry and Gravity for Weak Fields

1 Introduction

In this unit we will explore how gravitational waves are predicted by general relativity, and what some of their properties are likely to be. We will assume that the reader is only slightly familiar, at best, with the ideas and mathematical tools of general relativity. Thus we will begin with a description of accelerated motion in the context of special relativity as a way to start thinking about spacetime, tensors and other useful ideas. From there we will move on to a description of curvature and introduce some of the mathematical machinery needed to describe it. We will then make the connection between curvature and gravity, and outline how general relativity predicts the existence of gravitational waves as ripples in a flat background spacetime. Finally, we will give a brief description of some of the properties of the predicted waves and their sources.

The reader is assumed to have a background typical for someone with a Master's Degree in physics, but with no particular experience with general relativity. We expect some of the material to be challenging, but you should not feel that you have to understand everything in detail. We hope you will read this over with a view to getting some of the flavor for general relativity, without worrying too much about the details - though some details are provided for those interested. In particular, we do not intend any of this course to be a replacement for a course in general relativity. Rather, we wish to give the reader a basic feeling for how general relativity predicts the existence of gravitational waves, and what it says about the properties of those waves. Our hope is that the course will give participants a better intuitive understanding of relativity in general, and of general relativity and gravitational waves in particular, so that they can incorporate some of these ideas into their own classes.

2 Accelerated Motion in Flat Spacetime

As you learned in previous course materials, special relativity is described by a flat spacetime, somewhat analogous to a Euclidean space. In special relativity, spacetime distances are described by an interval given by

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \tag{1}$$

This interval is constant in all inertial reference frames, or in other words, in all frames that move at a constant velocity. So if we have two different frames of reference, one with some constant velocity relative to another, we can say that for all such pairs of frames

$$dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \tag{2}$$

Here we distinguish one frame's coordinates by placing a prime on them.

When we relax the condition of zero acceleration, the condition of constant interval is no longer valid. As an example, imagine that we are in a rocket ship speeding faster and faster through a totally flat Minkowski space. Imagine further that the acceleration felt by an observer in the frame is constant, *i.e.* that the acceleration *feels* constant to the observer[1].

How would such an acceleration look? The usual form of acceleration is $a = dv/dt$, but in relativity we know we do not have the usual definitions of velocity and time at our disposal since both will depend on who is doing the observing, and it is not clear that acceleration defined this way is at all constant to any observer.

One way around this problem is to imagine that we measure the acceleration from an inertial frame that instantaneously coincides with the frame of the accelerating frame. From that frame the effects of relativity will be small, or actually zero momentarily. As the rocket speeds away from one frame, we can switch to a different one that is in turn momentarily at rest with respect to the rocket's frame. By constantly switching our frame of reference to inertial frames that follow the rocket we can use our usual Newtonian ideas about speed, time and acceleration since the relative motion of the rocket and the currently coincident inertial frame are always small compared to the speed of light. The world line of the rocket and those of observers in successive inertial frames might look something like the diagram in Figure 1.

Notice that this definition of acceleration is similar to the definition of proper time, which is the time as measured by an observer at rest with respect to a particular clock. In this case, we call the *proper acceleration* the acceleration measured by an observer in an inertial frame that is instantaneously at rest with respect to the accelerating frame.

Now let's pick out a particular inertial frame that is coincident with our accelerating frame at an event A. See Figure 1. In this frame the line of coincidence connects back to the origin from the end of the red arc shown in Figure 2, with the area under the green

[1] Much of the following is based on Marolf (Marolf, 2003)

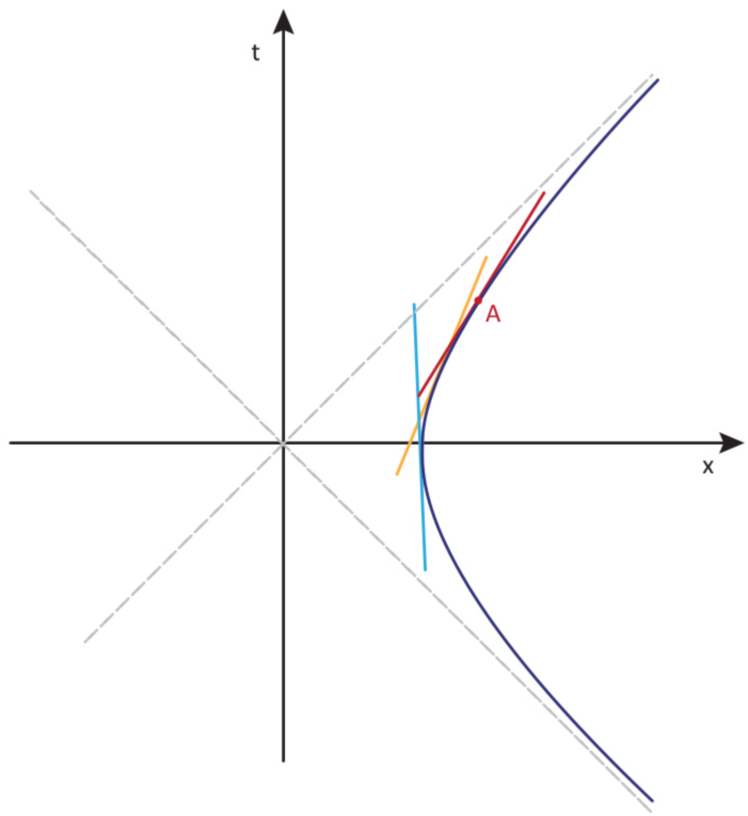

Figure 1: World line (curved) of an observer, initially moving to the left, but with a constant acceleration toward the right. The tangent lines to the path are world lines of inertial frames that are momentarily at rest with respect to the accelerating frame. Credit: SSU/A. Simonnet

line, above the x axis and to the left of the arc being $\phi/2$. Keep in mind that this frame is arbitrary, and that the diagrams in Figures 1 and 2 are valid for any inertial frame, and in particular it is valid for frames that are instantaneously coincident, or nearly coincident, with the accelerating frame. The position and time of the event in the x-t inertial system are

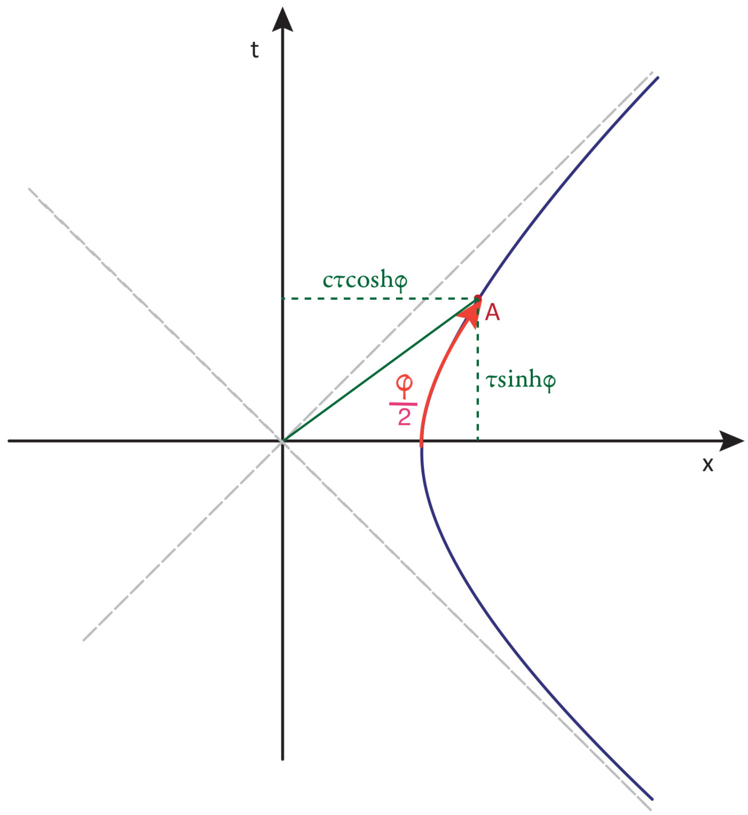

Figure 2: The x and t coordinates of an accelerating observer in the rest frame of another observer are related to the hyperbolic sine and cosine functions, each of which take an argument equal to the parameter, ϕ of the accelerated observer as seen by the unaccelerated one. The area of the wedge shape under the green line is $\phi/2$. Credit: SSU/A. Simonnet

$$x = c\tau \cosh \phi \tag{3}$$

$$t = \tau \sinh \phi \tag{4}$$

The velocity in this frame is $v = dx/dt$, which we can compute as follows:

$$dx = c\tau \sinh\phi \frac{d\phi}{dt} \tag{5}$$

$$dt = \tau \cosh\phi \frac{d\phi}{dt} \tag{6}$$

Taking the ratio of these we find that the velocity of the frame at the event A is given by

$$v = \frac{dx}{dt} = \frac{c\tau \sinh\phi \frac{d\phi}{dt}}{\tau \cosh\phi \frac{d\phi}{dt}} \tag{7}$$

Or, simplifying the expression,

$$v = c \tanh\phi \tag{8}$$

Now we can use our expression for the velocity to compute the acceleration by taking the time derivative. Since we are assuming we are in a coincident inertial frame, the usual Newtonian expression for the acceleration is valid and we have

$$\alpha = \frac{dv}{dt} = \frac{c}{\cosh\phi} \frac{d\phi}{dt} \tag{9}$$

We wish to evaluate this expression at the moment when the accelerating frames are coincident, in which case we have $dt = d\tau$, for proper time τ, and $\phi = 0$. Substituting we have

$$\alpha = \left(\frac{c}{\cosh\phi} \right)_{\phi=0} \frac{d\phi}{d\tau} \tag{10}$$

Or simply

$$\alpha = c \frac{d\phi}{d\tau} \tag{11}$$

This expression looks unremarkable, but in fact it says something quite interesting for accelerated motion in special relativity. Assuming that α is constant, we have

$$\frac{\alpha}{c} = \frac{d\phi}{d\tau} \tag{12}$$

or, after integrating,

$$\Delta\phi = \frac{\alpha}{c} \Delta\tau \tag{13}$$

So the parameter, ϕ, increases by equal steps for equal steps of proper time. Furthermore, ϕ tends toward infinity as the proper time does. This means that even accelerating at a constant rate forever, the velocity of an object never reaches or exceeds the speed of light. It has another interesting implication as well.

Homework Question 1

The relativistic boost parameter γ is generally written as

$$\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$$

How is this γ related to the parameter ϕ defined in Equation 8? Are the properties of ϕ consistent with the properties of γ as the velocity approaches the speed of light?

3 Maxwell's Equations and Spacetime

The previous section completes our very brief overview of special relativity. It was intended to give somewhat deeper understanding of spacetime than was developed in the *Big Ideas* materials. Before we move on to general relativity we want to take a moment to highlight one aspect of relativity that we have so far not given much attention to, but that is important both historically and scientifically. Namely, the connection between special relativity and Maxwell's equations.

We have seen how Maxwell's equations are related to the propagation of light, and we have made several comments about how they are consistent with special relativity without any need of modification. We wish to show this now, both to help us understand this connection and to introduce some of the mathematical machinery that will be needed to describe spacetime curvature in subsequent sections.

3.1 Four-vectors and the 3+1 View of Spacetime

To begin, we introduce the notion of a *four-vector*. The most basic four-vector is the one describing the position of some event in spacetime: $\mathbf{x} = (ct, x, y, z)$. If we wish to describe that event in a different inertial frame, then we would write the same vector with coordinates appropriate for the new system: $\mathbf{x}' = (ct', x', y', z')$. The connection between $\mathbf{x}$ and $\mathbf{x}'$ would be made by use of a transformation matrix, $\mathbf{\Lambda}$, such that

$$\mathbf{x}' = \mathbf{\Lambda}\mathbf{x} \tag{14}$$

If we wished to write this in terms of vector components, then we would write

$$x^{\mu'} = \sum_{\nu=0}^{\nu=3} \Lambda^{\mu'}{}_{\nu} x^{\nu} \tag{15}$$

Note here that the label indices run from 0 to 3, with $x_0 = ct$, $x_1 = x$, $x_2 = y$ and $x_3 = z$. Notice also that the index ν is the one summed over in the previous expression, and that it appears twice, once on the transformation matrix, $\mathbf{\Lambda}$, and once on the vector being transformed $\mathbf{x}$. To save ourselves having to always explicitly write the summation symbol, we will adopt a standard *summation convention* in which indices that appear twice, as ν does above, will be assumed to be summed over, the sum running from 0 to 3. So with this convention the previous equation becomes much more compact:

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \tag{16}$$

The transformation matrix must take account of any transformations that map one inertial frame into another one. These include simple axis rotations and translations as well as boosts, in which there is a non-zero relative velocity between the two frames. The rotation and translation aspects are just the familiar ones, so we will not worry about them. The boosts are the ones that are interesting in special relativity. For example, for two systems which have a relative velocity $\mathbf{V}$ directed along the x-axis, the transformation matrix has the form

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

where $\beta = v/c$ is the ratio of the relative speed of the two frames to the speed of light and γ is the boost parameter:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{18}$$

The transformation $\mathbf{\Lambda}$ is called the Lorentz transformation. It provides the rule for changing a four-vector from its components in one inertial frame to a different one.

Homework Question 2

A train has proper length L and lies along the x-axis. If the positions of the front and back of the train are x_f and x_b at some time t in its own frame, use the Lorentz transformation to find x'_f and x'_b, the positions of the ends of the train at some corresponding time t' in the frame in which the train is moving. Compare the positions of the front and the back in this frame, and show that the length of the train in the prime frame is that given by the standard Lorentz contraction factor. Assume the train moves along the x-direction relative to the prime frame.

There are many other four-vectors besides the four-position $\mathbf{x}$. For example, any vector that can be formed by multiplying the position four-vector by a scalar, or by taking its time derivative, is also a four-vector. These include the four-velocity (the derivative with respect to proper time of the position vector) and the four-momentum. The thing to keep in mind about four-vectors, and the thing that makes them so useful, is that they are invariant under Lorentz transformations. We will make great use of them, and related objects, in much of what we discuss from now on.

Aside from transforming four-vectors between different inertial frames, we can also manipulate them in the usual ways we do vectors. For instance, we can find the dot product between two of them. However, there are some rules about doing this that we should learn now because they will be necessary later. For instance, you might have noticed that in the vector transformations we did earlier, one of the vectors had a superscript index and the other had a subscript. In general, when we combine vectors we must do it this way, with an upper and a lower. In special relativity, at the very least, it introduces a sign change in the time component, and it becomes essential later when we are dealing with curved spaces in general relativity.

The way we raise an index is by operating on it with the *metric tensor*, η:

$$x^\mu = \eta^{\mu\nu} x_\nu \tag{19}$$

,or conversely

$$x_\mu = \eta_{\mu\nu} x^\nu \tag{20}$$

Notice that the dummy index matches the vector we are raising or lowering, and the remaining index matches the resulting vector. These are not different vectors, they are exactly the same; their components can have small differences, as we'll see below. The one with the upper index is called the contravariant component of the vector. The one with the lower index is called the covariant component, or sometimes a *one-form*. We don't really have to worry about the differences between these two forms in special relativity, but it becomes important in general relativity, and so we introduce it now.

So if we want to take the dot product of two four-vectors we must use the metric tensor, or just metric for short. In special relativity the metric is simple. It looks like this:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

You will notice that the metric is almost as we would expect in a four-dimensional Euclidean space, but it has a negative $(0, 0)$ element - this is not a typo. The metric of Minkowski spacetime differs from our familiar Euclidean space in that it must keep the speed of light constant. The way it does this is by treating the time element with an opposite algebraic sign than it does the space dimensions. That allows the interval, ds^2 in special relativity to be positive, negative or zero. In Euclidean space, distances between objects can only be positive or zero; there is no such thing as a "negative distance" (as opposed to displacement) between two objects. Spacetime is quite different.

In spacetime, events (as opposed to objects) that have intervals that are negative, also called *timelike*, have a time separation that is bigger than their space separation. That means it is possible for an observer to travel from one event to the other. Events separated by a positive interval, called *spacelike* have a space separation that is larger than their time separation. That means that an observer would have to travel faster than the speed of light in order to be at both events, but such observers are not allowed. For events with a spacelike separation even light is too slow to bridge that gap. When two events have a null separation their time parts and space parts are exactly the same, and so they cancel out. Photons and other objects that move with the speed of light have null separations, and any object with a null four-velocity is moving at the speed of light. It is the negative time-time (η_{00}) element that allows for this strange-seeming behavior in special relativity.

For the Minkowski metric, the only difference between the contravariant form of a vector and its covariant form is that the time components differ in algebraic sign. We can use the metric to compute the dot product as follows:

$$\mathbf{A} \cdot \mathbf{B} = A^{\mu} B_{\mu} = A^{\mu} \eta_{\mu\nu} B^{\nu} \tag{22}$$

In particular, if we wish to write out the spacetime interval, we can do it this way

$$ds^2 = \mathbf{dx} \cdot \mathbf{dx} = dx^{\mu} dx_{\mu} = dx^{\mu} \eta_{\mu\nu} dx^{\nu} \tag{23}$$

When we write this all out explicitly, noting that the off-diagonal terms are zero, we have

$$ds^2 = dx^0 dx_0(-1) + dx^1 dx_1(1) + dx^2 dx_2(1) + dx^3 dx_3(1) \tag{24}$$
$$= -cdt^2 + dx^2 + dy^2 + dz^2 \tag{25}$$

In spacetime we generally use ds^2 for the interval to distinguish it from the distance dr^2 in space. Also, we don't ever use ds itself since it would sometimes be imaginary. It seems better simply to think about the square of the interval, and refer to it as the interval or the interval squared, interchangeably.

Homework Question 3

A four-vector $A_\mu = (A_0, A_1, A_2, A_3)$, with $(0, 1, 2, 3)$ corresponding to (ct, x, y, z). A second vector is $B_\mu = (B_0, B_1, B_2, B_3)$. Use the metric tensor $\eta^{\mu\nu}$ to find the components of $A^\mu = \eta^{\mu\nu} A_\nu$ and $B^\mu = \eta^{\mu\nu} B_\nu$. And then find the dot product of the two vectors, $A^\mu B_\mu$. How is it that the order in which you multiply the vectors does not matter? Or in other words, why is it that $A^\mu B_\mu = B^\mu A_\mu$?

3.2 Maxwell's Equations in 3+1 Spacetime

As we saw earlier, Maxwell's equations consist of four partial differential equations that describe the electric and magnetic fields and their coupling to one another. In MKS units they are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{26}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{27}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{28}$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \tag{29}$$

These can be written more compactly in 3+1 dimensions. We have, in the more customary Gaussian units where

$$\epsilon_0 \to \frac{1}{4\pi}$$

and

$$\mu_0 \to \frac{4\pi}{c}$$

Maxwell's equations become

$$\frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = \frac{4\pi}{c} j^{\nu} \tag{30}$$

and

$$\frac{\partial F^{\mu\nu}}{\partial x^{\alpha}} + \frac{\partial F^{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial F^{\alpha\mu}}{\partial x^{\nu}} = 0 \tag{31}$$

with the four-vector current density j^{ν} given by

$$j^{\nu} \equiv (c\rho, j^{x}, j^{y}, j^{z}) \tag{32}$$

The object $F^{\mu\nu}$ is written as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^{x} & -E^{y} & -E^{z} \\ E^{x} & 0 & -B^{z} & B^{y} \\ E^{y} & B^{z} & 0 & -B^{x} \\ E^{z} & -B^{y} & B^{x} & 0 \end{pmatrix} \tag{33}$$

These equations are equivalent to equations 26 - 29, which can be shown by substituting the definitions of $F^{\mu\nu}$ and j^{ν} into equations 30 and 31.

Homework Question 4

Try this substitution for a couple of the equations if you haven't done it before.

Objects like $F^{\mu\nu}$ are called *tensors* - actually, four-vectors and scalars are tensors as well, and Maxwell's equations are called *tensor equations*. Any equation that is a four-tensor equation is invariant under Lorentz transformations. In fact, the postulate of relativity is that all the laws of physics must be invariant under Lorentz transformations, or in other words, that all the laws of physics must be four-tensor equations. Maxwell's equations satisfy this postulate automatically. See the Math Supplement if you would like to see how to prove that the Maxwell equations are Lorentz invariant.

There is one interesting note about Maxwell's equations. As we have already seen, they predict electromagnetic waves, and the speed of these waves, c is the product of two constants:

$$c = (\mu_0 \epsilon_0)^{-\frac{1}{2}} \tag{34}$$

These two constants are invariant under Lorentz transformations since they are just numbers. This suggests that the speed of light is also constant. If we cast the Maxwell equations in the form of a wave equation, we see that the only velocity in them is the velocity of the waves themselves, which is the product of the two aforementioned physical constants. From this simple fact we can see that Maxwell's equations point the way to special relativity. Albert Einstein had the Maxwell equations in mind when he was thinking about his theory in its earliest stages. He intuitively realized that an observer moving along with the waves would not see them oscillate. Instead the fields would be frozen in a sine wave pattern. This would violate the predictions of Maxwell's equations, and so Einstein deduced that such an observer could not exist. As often happened, his intuition was correct.

4 Curvature

Our attention so far has all been on flat spaces: the Minkowski space in which special relativity unfolds is a flat space, just like the Euclidean space to which we are accustomed. Except for their dimensionality and the sign difference between the spacial and temporal parts (a BIG difference) these two spaces are the same. In this section we will begin to think about curved spaces, and we have to consider this question: What is the difference between accelerated motion viewed from a flat space and non-accelerated motion viewed from a space that is curved? And furthermore, what do we even mean by "a space that is curved?" Those questions will propel us to a viewpoint that gravity is curvature, the viewpoint we will find in general relativity. We will not be exhaustive in our look into curvature, and anyone who would like to delve more deeply into this should read through some of the very good introductory texts on general relativity, which we will list at the end of the section.

To start, lets think about curvature that we are most used to. We have already looked at this in the *Big Ideas* materials, but here we will be a little more mathematical in how we approach it.

The easiest way to think about a curved space is to use one that we are already deeply acquainted with, that of a spherical surface. We know that the distance between any two points on the surface of a sphere is related to the distance interval on the surface:

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \tag{35}$$

where θ and ϕ are the polar and azimuthal angles, respectively (similar to latitude and longitude on Earth) and r is the sphere's radius of curvature, as in Figure 3. We also know that we can write the square of the distance element on the surface as a product of the coordinate distances and the metric tensor, here called g to distinguish it from η, the metric of special relativity, as follows:

$$
\begin{aligned}
ds^2 &= g_{ij} dx^i dx^j && (36)\\
&= g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 + g_{21} dx^2 dx^1 && (37)\\
&= g_{11} d\theta d\theta + g_{12} d\theta d\phi + g_{22} d\phi d\phi + g_{21} d\phi d\theta && (38)\\
&= g_{11} d\theta^2 + g_{12} d\theta d\phi + g_{22} d\phi^2 + g_{21} d\phi d\theta && (39)
\end{aligned}
$$

with i and j take on values 1 (θ) and 2 (ϕ) and we use the standard summation convention for repeated indices. Comparing this expression (40) to Equation 36 we see that

$$
\begin{aligned}
g_{11} &= r^2 && (40)\\
g_{22} &= r^2 \sin^2(\theta) && (41)\\
g_{12} &= g_{12} = 0 && (42)
\end{aligned}
$$

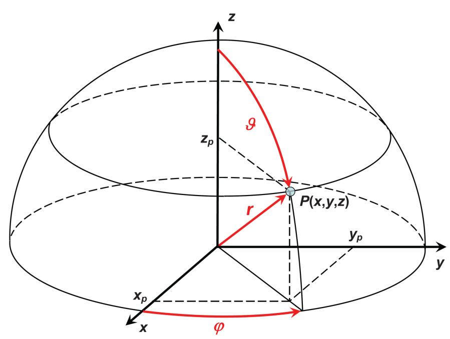

Figure 3: The 2-d coordinates used to describe the position on the surface of a sphere are the angles θ, that measures the distance from the positive z-axis, and ϕ, that measures the distance around the z-axis from some reference. The radius of the sphere r is not a coordinate, it merely serves as a parameter to scale the sphere whose surface comprises our space. The image also shows how the xyz Cartesian coordinates are related to the spherical coordinates. Credit: `http://www.seos-project.eu/modules/laser-rs/laser-rs-c03-s01-p01.html`

The thing to notice about this metric is that some of its elements depend on position; the g_{22} term in particular depends on θ. When the metric of a space depends on the position that can be a signature of curvature. However, having a variable metric is not enough to say that a space is curved, only that the coordinates that we are using are curved.

As another example, consider a flat Euclidian space with a metric given by

$$g_{11} = 1 \tag{43}$$
$$g_{22} = 1 \tag{44}$$
$$g_{12} = -\cos\theta \tag{45}$$
$$g_{21} = -\cos\theta \tag{46}$$

and coordinate $x^1 = q$ and $x^2 = p$. The length interval squared corresponding to this metric is

$$ds^2 = dq^2 + dp^2 - 2dqdp\cos\theta \tag{47}$$

In this interval we see the familiar law of cosines, which provides the distance between the ends of two line segments that make an angle θ. This is usually presented as a way to compute the third side of a triangle when two sides, along with the angle between them, are already known. But it can also be thought of as the square of the distance element for an oblique coordinate system such as the one shown in Figure 4. In that system the angle θ is not one of the coordinates, it is merely a parameter that tells us the amount by which the coordinate axes are skewed - in this regard it plays a role similar to that of the radius r in the spherical coordinates we described previously.

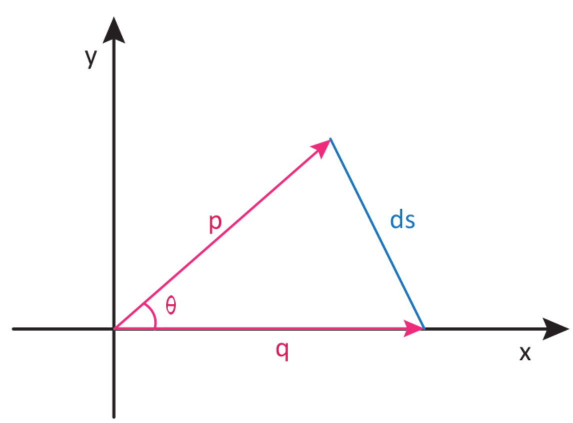

Figure 4: In an oblique coordinate system, in which the two coordinate axes are not perpendicular, the distance interval depends on a cross term between the two coordinates. This is reflected in the metric of Equation 48. Credit: SSU/A. Simonnet

These coordinates are perfectly adequate to describe a flat Euclidean space, though they are usually more trouble to use than the standard perpendicular xy coordinates. However, there are systems where these sorts of oblique coordinates are convenient. An example would be to use them to describe the oblique orientations of some crystals. In those cases the individual p and q coordinate axes do not even have to have the same scale since the atomic spacings along the different directions can be different. We won't bother anymore with such coordinates, they merely give us a nice example of how the metric of a space can be used to determine the distance interval in that space. By the way, notice that when θ is 90 degrees we recover the usual xy cartesian system.

As another example, we can describe a plane with plane-polar coordinates. In these coordinates the position of any point in the x-y plane is given by an angle, θ, usually measured counterclockwise around the vertical direction from the positive x-axis, and the distance, r, from the center of rotation. This is clearly a flat space, and yet the metric depends on position, as you will show in the following homework.

Homework Question 5

Find the distance interval ds^2 for a plane-polar coordinate system, and from that find the metric tensor using

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

You will use this result in Homework Question 4, below.

You should find that the metric in the previous homework depends on position. And yet, it describes a flat space. We have two different metrics that describe exactly the same space, which in this case is flat. One of them has constant metric coefficients, the other, like the metric for our spherical 2-d surface, does not. This is a general rule about the relationship between geometry and metrics: the metric certainly encodes the geometry of the space it describes, but not generally in a unique way or even obvious way. There can be many (infinitely many - how many values can θ take on for the oblique coordinates?) metrics that can describe the same space, but they must all have the intrinsic geometry of that space.

You might object that our spherical surface coordinates are not actually describing a curved space. They are describing a curved 2-dimensional surface in an otherwise flat 3-dimensional space. You would be correct to make such an objection, at least in part. However, if we consider only the curved surface itself, then the metric is describing everything about the geometry of that space. We don't have to appeal to a higher-dimensional space in order to make sense of what the metric tells us. And we don't have to think of the radius, r, as anything other than a parameter that tells us how big things are. We are used to thinking of r as the actual radius of an actual sphere, but it need not be. We will find that the same is true in general relativity when we consider the curvature of 3-space or 3+1 space: we can imagine those spaces embedded in higher dimensional spaces if we want to, but it is really no help to do so. We cannot visualize or measure any higher dimension spaces, and in any event, the metric will tell us everything we need to know about the curvature of the spaces we can measure.

Our description of curvature is still not quite rigorous. It is true that on the flat surfaces we have described, the metric describing those surfaces gives us a hint about their curvature depending on whether the metric used for them had constant components or not. However, as we see from the examples mentioned, we cannot completely rely on the metric components to tell us about the curvature of a space. For example, in three dimensions, we can use spherical polar coordinates. In such coordinates the distance interval is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{48}$$

,for the same definition of r, θ and ϕ as we used before. Clearly the metric components depend on the coordinates, and yet this metric is describing a flat 3D space. The variation of the metric here is simply an artifact of the coordinate system we are using. To get a better idea of what curvature really is we need to use a little more mathematical sophistication.

One way to think about a curved surface that does not involve the metric is to consider how the space affects the *parallel transport* of a vector. The notion of parallel transport involves moving a vector around in the space while keeping it always parallel to itself, or at least, keeping it as parallel to itself as possible. If you imagine performing this operation on some vector $\mathbf{V}$ in a flat space, then neither the vector nor its components will change at all as it is moved around. If at the end of its transport it arrives back where it started, it will be exactly the same as when it left. Mathematically we could say that

$$\mathbf{V}_{final} - \mathbf{V}_{initial} = 0 \tag{49}$$

On the other hand, something completely different happens in a curved space. Imagine we are in a 2-dimensional spherical space, as shown in Figure 5. If we start with a vector that sits on the equator of the sphere and that points in a direction perpendicular to the equator, then we can imagine parallel transporting it by moving it toward one of the poles along a line of constant azimuthal angle, ϕ, to which it will remain parallel as we move away from the equator. Upon reaching the pole we can transport the vector parallel to itself, and also parallel to another line of constant ϕ back toward the equator. We might arrive at the equator 90 degrees away from our starting position, but now with the vector parallel to the equator. If we then move the vector back to our starting point, always keeping it parallel to itself, and therefore parallel to the equator, we will find that the final position of the vector will be rotated 90 degrees with respect to how it started: If our journey had been on the surface of Earth with the vector initially pointing north, we might end up at the same point with the vector pointing east.

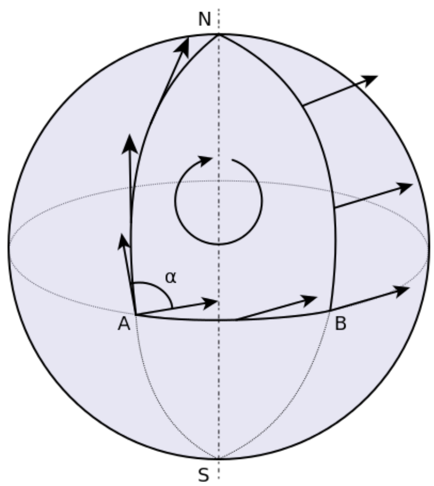

Figure 5: Parallel transporting a vector around a closed path surrounding a region on a curved space does not leave it unchanged. In the example shown for a spherical surface, transporting a vector from the equator to the pole, then back to the equator, and finally moving back to its original position along the equator, will rotate the vector by some angle, $\alpha \neq 0$. Credit: Wikipedia

Mathematically we could approach this process as follows (based on treatment in Schutz (2009)): Imagine the vector moves from some initial position (I) to some final position (F) along a path s and see how the vector changes. We could write this as:

$$\delta \mathbf{V} = \mathbf{V_F} - \mathbf{V_I} \tag{50}$$

$$= \int_I^F \frac{d\mathbf{V}}{ds} ds \tag{51}$$

In a flat space we may write

$$\delta \mathbf{V} = \int_I^F \frac{\partial \mathbf{V}}{\partial s} ds \tag{52}$$

But in a curved space, the above expression is incomplete because the coordinate basis vectors can change as we move the vector. To write the derivative of a vector we must take this change into account:

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\mu}{\partial x^\beta}\tilde{\mathbf{e}}_\mu + V^\mu \frac{\partial \tilde{\mathbf{e}}_\mu}{\partial x^\beta} \tag{53}$$

The first term on the right-hand side of this equation reflects the change in the vector components V^μ itself, and the second term represents the change in the basis vectors $\vec{e}_\mu$ of the coordinates. This derivative is generally written in a slightly different way. If we write

$$\frac{\partial \tilde{\mathbf{e}}_\mu}{\partial x^\nu} = \Gamma^\alpha{}_{\mu\nu}\tilde{\mathbf{e}}_\alpha \tag{54}$$

Then we can rewrite Equation 54 as

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\tilde{\mathbf{e}}_\alpha + V^\mu \Gamma^\alpha{}_{\mu\beta}\tilde{\mathbf{e}}_\alpha \tag{55}$$

$$= \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}\right)\tilde{\mathbf{e}}_\alpha \tag{56}$$

The term in the parenthesis takes into account both the change in the vector components as we move around in the space, and also the change in the coordinate basis vectors. It is called the *covariant derivative* of the vector $\mathbf{V}$. The terms $\Gamma^\alpha_{\beta\gamma}$ are called *Christoffel symbols*: **They take into account how the coordinate basis vectors change as we move around in the space**. The way to think of them is that $\Gamma^\alpha_{\beta\lambda}$ is the α^{th} component of the change of the β^{th} basis vector with respect to the λ^{th} coordinate. In a flat space with constant basis vectors, the Christoffel symbols are zero. In curvilinear coordinate systems they are not zero, even if the space described is flat. For intrinsically curved spaces they are not zero either.

To parallel transport a vector is to transport it in such a way that its covariant derivative vanishes. Mathematically we may write

$$\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta} = 0 \tag{57}$$

Parallel transport does not mean that the vector will necessarily end up parallel to its initial position. That will only be true in a flat space. In more general spaces the notion that a vector in one part of the space is parallel to a vector in another part of the space might not even make sense. For instance, what does it mean to say that two displacement vectors along Earth's surface, one at the equator and the other at the south pole, are parallel to each other? In a curved space, the closest we can get to parallel is parallel transport.

So getting back to the parallel transport of the vector $\mathbf{V}$, what we would like to know is how the vector changes as we parallel transport it around the space. Specifically, we would like to know how the vector changes if we transport it around a closed path where our final point coincides with our starting point.

We imagine that the path of the vector is made up of short segments along two independent coordinates of our space, call them x^1 and x^2, something like the situation shown in Figure 6. Furthermore, we can imagine that x^1 ranges from an initial value a to a slightly different value $a + \delta a$. Similarly we imagine that x^2 ranges between values b and $b + \delta b$. So if we compute the change in our vector along two points on our path, say along a constant value $x^1 = a$, the change in the vector is

Homework Question 6

The Christoffel symbols $\Gamma^{\gamma}_{\beta\mu}$ introduced above can be found from the metric components of a space using the following expression:

$$\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right)$$

The comma indicates that a partial derivative is to be taken with respect to that index:

$$V^{\mu\nu}{}_{,\beta} \equiv \frac{\partial V^{\mu\nu}}{\partial x^{\beta}}$$

Use the expression above for $\Gamma^{\gamma}{}_{\beta\mu}$, and your result from Homework Question 4, to compute the Christoffel symbols for a plane-polar coordinate system.

Remember, in a 2-d space the indices only run from 1 to 2. Also, the Christoffel symbols are symmetric in their lower indices, and the metric for this coordinate system only has one element that is not constant. Given these facts, you won't have many terms to calculate.

What do you think the Christoffel symbols will be for the metric in Equation 48? Explain.

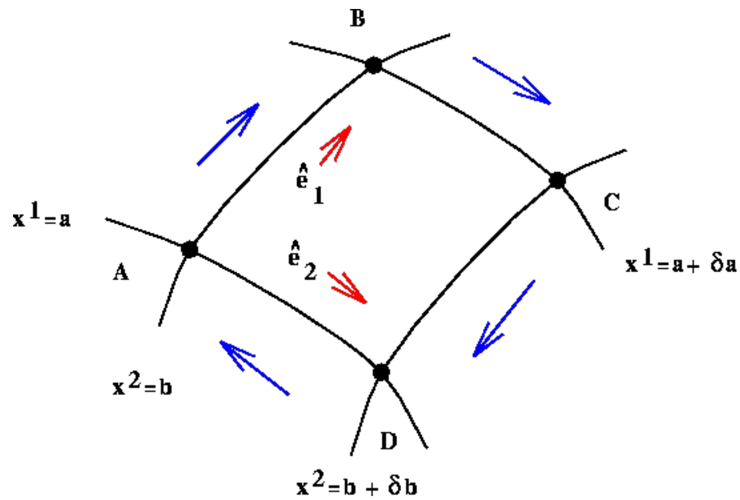

Figure 6: A possible path around which to parallel transport a vector. Credit: http://www.mth.uct.ac.za/omei/gr/chap6/node9.html.

$$\delta V^{\alpha}_{x^1=a} = \int_{x^1=a} \frac{\partial V^{\alpha}}{\partial x^2} dx^2 \tag{58}$$

Because we parallel transport this vector, we know that its covariant derivative is zero. That means we can replace the integrand using Equation 58.

$$\delta V^{\alpha}_{x^1=a} = -\int_{x^1=a} \Gamma^{\alpha}{}_{\nu 2} V^{\nu} dx^2 \tag{59}$$

We have four segments that must be traversed to bring the vector back to its starting point. After summing them all up (see Math Supplement for details) the result is

$$V^{\alpha}_{final} - V^{\alpha}_{initial} = \delta b \delta a \left[\frac{\partial}{\partial x^{\lambda}} \Gamma^{\alpha}{}_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}} \Gamma^{\alpha}{}_{\nu\lambda} + \Gamma^{\alpha}{}_{\nu\lambda} \Gamma^{\nu}{}_{\mu\sigma} - \Gamma^{\alpha}{}_{\nu\sigma} \Gamma^{\nu}{}_{\mu\lambda} \right] V^{\nu} \tag{60}$$

$$= R^{\alpha}{}_{\nu\sigma\lambda} V^{\nu} \delta b \delta a \tag{61}$$

The term in the square brackets is called the Riemann curvature tensor, $R^{\alpha}_{\nu\sigma\lambda}$, or just **R**. It depends on the Christoffel symbols and their derivatives, which means it depends upon the first and second derivatives of the coordinate basis vectors. In a flat space **R** will be zero, though individual Christoffel symbols and their derivatives might not. In fact, the condition for a curved space is $\mathbf{R} \neq 0$.

Note that the Riemann tensor is a fourth-rank tensor because it has four indices. In regular three-space it will be a 3-dimensional tensor, with each of its indices running from 1 to 3. In a 2-dimensional space, like the surface of a sphere, it is 2-dimensional, with each index running from 1 to 2. In spacetime the Riemann tensor is a 4-dimensional tensor (3 space, 1 time). Its indices take on values from 0 to 3. In all these spaces it is fourth rank. Remember that a tensor's rank is how many indices it carries. Its dimensionality, on the other hand, matches the dimensionality of the space in which it lives.

The importance of the Riemann curvature tensor is its connection to the metric tensor of a space. Both must have the same dimensionality, but they do not have the same rank. The metric is rank 2, not rank 4. We can show (but won't) that

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial^2 g_{\sigma\nu}}{\partial x^\beta \partial x^\mu} - \frac{\partial^2 g_{\sigma\mu}}{\partial x^\beta \partial x^\nu} + \frac{\partial^2 g_{\beta\mu}}{\partial x^\sigma \partial x^\nu} - \frac{\partial^2 g_{\beta\nu}}{\partial x^\sigma \partial x^\mu} \right) \tag{62}$$

This notation is a bit cumbersome, so we will instead use a compact form that emphasizes the tensor nature of $R^{\alpha}_{\beta\mu\nu}$ and allows us to see its internal symmetries more easily. Instead of writing out the partial derivatives, we imply them by using a comma in the index list. We differentiate with respect to indices following the comma:

$$F_{\alpha,\beta} \equiv \frac{\partial F_\alpha}{\partial x^\beta} \tag{63}$$

Using this shorthand notation, Equation 63 is

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right) \tag{64}$$

Now we can easily see that $R^{\alpha}_{\beta\mu\nu}$ is antisymmetric under exchange of its first and third indices, as well as its second and fourth. In addition, it is symmetric under exchange of its first pair of indices with its second pair. Given its symmetries, $\mathbf{R}$ is much less complicated than it looks: in four dimensions the number of independent components of $\mathbf{R}$ is "only" 20. The Riemann tensor allows us to connect the curvature of spacetime to gravity.

To put all of this into perspective, it is good to remember that our definition of curvature depends upon the idea of parallel transport. This is key: we know that **in a flat space a vector will be unchanged if moved around in the space while being kept parallel**. On the other hand, **in a curved space there is generally no way to move a vector around and keep it parallel; it will be modified as it moves around**, even if we keep it "locally parallel." This modification will happen in such a way that **if we bring the vector back to where we started it will no longer be the same vector we had when we started**. Our ability to deal with this concept of parallel transport depended on taking the **covariant derivative** of a vector, a derivative that takes into consideration not only changes of the vector components, **but also changes to the coordinate basis vectors**. We found that in a general coordinate system, in which the basis vectors are not fixed, it is their turns and twists that describe the curvature of a space. Of course, if we choose a small enough region, then we can approximate the space as flat, at least to some degree. This is similar to approximating a region in the world line of an accelerating object in flat space with a momentarily corresponding inertial frame. Just as in that case, we can approximate any point in space (assuming it is continuous and differentiable) with a locally flat tangent plane. That does not, however, change the globally curved nature of the space, and the Riemann tensor picks out this curvature for us.

> **Homework Question 7**
>
> Imagine performing the parallel transport of a vector around the surface of a sphere, but in 3-space, not 2-space, and using a 3-vector, not a 2-vector. That is, perform the transport in the three dimensional space in which the spherical surface is embedded, not in the surface itself. Does the vector have to point along the surface for this situation, or can it point in any of the three directions? What will be the result of this parallel transport? Will the vector be changed or not if you make a loop that returns the vector to its starting point?
>
> If you described this space with a spherical-polar coordinate system, would the Christoffel symbols be zero or non-zero? How does their value relate to your answer about the parallel transport of the vector? Or in other words, are Christoffel symbols alone enough to describe the curvature of a space, as opposed to the curvature of the coordinate system used to describe that space?

In the next section we will see how curvature is related to the distribution of mass-energy, and how a gravitational field arrises from that distribution.

5 The Einstein Field Equations

The first step in writing the field equations for gravity is to realize that gravity in general relativity will be related to the distribution of mass, just as in Newtonian gravity. However, in relativity it is not just rest mass that contributes. We must also include other sources of energy like kinetic energy and potential energy - even the energy of gravity itself contributes.[2] The object that describes the distribution of mass-energy in relativity is

[2] This makes general relativity a non-linear theory, and the equations that describe it will themselves be non-linear. Therefore, they are much harder to solve than the equations describing Newtonian gravity.

the *stress-energy tensor*, denoted as $T^{\alpha\beta}$. It is a second rank, symmetric tensor with the following general properties:

$$\begin{array}{lcl} T^{00} & \rightarrow & \text{energy density} \\ T^{0j} & \rightarrow & j \text{ momentum density} = \text{energy flux through surface of constant } x^j \\ T^{j0} & \rightarrow & \text{energy flux through surface of constant } x^j = j \text{ momentum density} \\ T^{ij} & \rightarrow & \text{stress tensor} = i \text{ momentum flux density in } j \text{ direction} \end{array}$$

with i and j running from 1 to 3, the usual 3-dimensional space indices.

The stress-energy tensor tells us all there is to know about the distribution of mass and energy in a region. So we should be able to use it to express the conservation laws for energy and momentum in a straightforward way. This is indeed the case. For instance, consider a small cube in which the sides of length δ are aligned with the usual Cartesian coordinate axes. We can express the energy flowing into and out of the sides of the cube in terms of the components of $T^{\alpha\beta}$ as follows:

$$F_x = (T^{01}_{x_1} - T^{01}_{x_2})\delta^2 \tag{65}$$

$$F_y = (T^{02}_{y_1} - T^{02}_{y_2})\delta^2 \tag{66}$$

$$F_z = (T^{03}_{z_1} - T^{03}_{z_2})\delta^2 \tag{67}$$

where $x_2 = x_1 + \delta$, and F_{x_i} represents the flow of energy across the surface at some particular value of x_i, with same being true for the y and z expressions.

We know that the sum total of the energy flowing into/out of the cube must be balanced by the change in the energy content of the cube. The latter is the product of the energy density, T^{00}, and the volume of the cube, δ^3. Setting these equal we have

$$\delta^3 \frac{\partial T^{00}}{\partial t} = F_x + F_y + F_z \tag{68}$$

$$= (T^{01}_{x_1} - T^{01}_{x_2} + T^{02}_{y_1} - T^{02}_{y_2} + T^{03}_{z_1} - T^{03}_{z_2})\delta^2 \tag{69}$$

If we divide by δ^3 and then take the limit as $\delta \rightarrow 0$, we get

$$\frac{\partial T^{00}}{\partial t} = \frac{\partial T^{01}}{\partial x} + \frac{\partial T^{02}}{\partial y} + \frac{\partial T^{03}}{\partial z} \tag{70}$$

Equation 71 expresses the conservation of energy and is often called the *continuity equation*, particularly in the study of fluids. It can be expressed much more compactly as

$$T^{\alpha\beta}{}_{,\beta} \equiv \frac{\partial T^{\alpha\beta}}{\partial x_\beta} = 0 \tag{71}$$

We must remember that the time part of the metric (η_{00}) imparts a negative sign to the time part of Equation 72 in order to see the equivalence between it and 71.

The stress-energy tensor describes the distribution of mass and energy in space, which will be the source of gravitational curvature. That means we must find a second rank symmetric tensor that describes the curvature of spacetime, and then somehow match the two. We can make such a tensor via a contraction of the Riemann tensor. It turns out that a contraction on its first and third indices is the most general way to get a symmetric second rank tensor out of the Riemann tensor, and doing so gives us the *Ricci tensor*:

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} \tag{72}$$

The Ricci tensor cuts the number of independent elements from 20 to 10. We can further contract the Ricci tensor to obtain the Ricci scalar, R:

$$R \equiv R^{\gamma}_{\gamma} = g^{\gamma\mu} R_{\gamma\mu} \tag{73}$$

Combining the Ricci scalar with the metric tensor gives us another, independent, second rank symmetric tensor. Combining these two we have a tensor that we can match to the stress-energy tensor:

$$R_{\alpha\beta} + kRg_{\alpha\beta} = CT_{\alpha\beta} \tag{74}$$

where k and C are constants to be determined. By insisting that the equation be consistent with Newtonian gravity in the case of weak fields and that it also be consistent with conservation of energy ($T^{\alpha\beta}{}_{,\beta} = 0$) we can show that $k = -1/2$ and $C = 8\pi G/c^4$. We can then write a theory of gravity as curvature:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \tag{75}$$

This equation is called the *Einstein equation*. It describes the relationship between the mass-energy distribution (on the right-hand side) and the curvature of spacetime (on the left-hand side). It is generally written in terms of the Einstein tensor, $G_{\alpha\beta}$,

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \tag{76}$$

It was first published by Albert Einstein in 1915 after a decade of work, and is the fundamental equation of his General Theory of Relativity.

Since there are ten independent terms in the tensors $T^{\alpha\beta}$ and $G^{\alpha\beta}$, the Einstein "equation" is really a shorthand for ten Einstein "equations," just as the vector form of Newton's Second Law is really a shorthand for three independent equations - one for each spatial dimension. From the Einstein equation can be deduced the spacetime curvature around stars (the Schwarzschild spacetime metric, or the Kerr metric, depending upon whether the star is rotating or not) and the Friedmann-Robertson-Walker metric that describes the dynamics of the universe.

Interestingly, the expanding solution to his equation did not occur to Einstein, and his equation as written above implies that the universe cannot be static. To counteract an implied collapse Einstein introduced a term $\Lambda g_{\alpha\beta}$ to stabilize the universe; he called Λ the *cosmological constant*. Einstein regretted the addition when Hubble discovered that the universe expands, and the term was abandoned for many decades. Recently it has been resurrected because astronomers have found the universe to be expanding faster over time, just the effect the Λ term would create.

We will not spend any time exploring the well-known solutions to the Einstein equations for stars and cosmology. The interested reader is encouraged to look at the textbooks listed at the end of this document for further studies, or in the many other very good books that have been written about cosmology and black holes. We will now (finally) turn our attention to the topic for this course, gravitational waves, and their relation to the Einstein equation and general relativity.

6 Gravitational Waves in Weak Fields

The Einstein equations are generally quite difficult to solve. The cases mentioned previously for stars and cosmology are two of the few cases where symmetry allows an analytic solution of the equations. Usually it is necessary to employ a computer and solve the equations numerically. This is especially true in the case of strong fields, where the non-linearity of the equations comes into play and it becomes necessary to compute the gravitational curvature caused by the gravitational curvature! Many of the sources expected to create gravitational waves fall into this latter category, since the emission of strong gravitational waves requires very strong fields that are highly time dependent.

Happily, we do not have to worry about the details of how the waves are generated if all we want to do is study their properties at some great distance from their source, like those at which the LIGO (and other) gravitational wave detectors hope to detect them. So we will look at the properties of the waves when they have traveled far and lost much of their strength. This *weak field* case can be studied analytically, and we will end this part of the course with a brief overview of it.

To begin with, consider a region of space far away from any strong gravitational fields. In such a region the spacetime curvature will be quite small, approximating that of the flat Minkowski space of special relativity. Under these conditions we can approximate the spacetime metric g as being the Minkowski metric η with a small perturbation, which we will call h:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{77}$$

with $|h| << 1$.

6.1 Linearized Gravity

We are now ready to linearize the gravitational field equations and find their solutions under these restrictions. We will begin with the Riemann tensor written in terms of the metric:

$$R_{\alpha\beta\mu\nu} \equiv g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} - g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{78}$$

$$= \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} - h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right) \tag{79}$$

We get the Equation 80 by substituting Equation 78 into Equation 79. To first order only the h terms in the metric contribute to curvature; η certainly does not, and so only the h parts remain. To find the Ricci tensor, contract on α and μ to get

$$R_{\mu\nu} = \frac{1}{2}\left(h^{\gamma}_{\mu,\gamma\nu} + h^{\gamma}_{\nu,\gamma\mu} - h_{,\mu\nu} - \Box h_{\mu\nu}\right) \tag{80}$$

Here we have used

$$h \equiv h^{\alpha}{}_{\alpha} \tag{81}$$

and for arbitrary function, f:

$$\Box f \equiv f^{,\mu}{}_{,\mu} = -\frac{\partial^2 f}{\partial t^2} + \nabla^2 f \tag{82}$$

to write Equation 81 more compactly. We can rearrange the terms in Equation 81 to obtain the Ricci tensor is a slightly different form...

$$R_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + \frac{\partial}{\partial x^{\mu}}\left(h^{\gamma}_{\nu,\gamma} - \frac{1}{2}h_{,\nu}\right) + \frac{\partial}{\partial x^{\nu}}\left(h^{\gamma}_{\mu,\gamma} - \frac{1}{2}h_{,\mu}\right)\right] \tag{83}$$

This expression would be much simpler if the terms inside the parentheses would vanish. Fortunately, we still have four degrees of freedom in our choice of coordinates, and since the choice of coordinates is completely arbitrary, we are free to use any we like (cf: the gauge conditions from Equation 72 in the Math Supplement). We can make the terms in the parentheses vanish if we choose coordinates in which

$$h^{\gamma}{}_{\nu,\gamma} - \frac{1}{2}h_{,\nu} = 0 \tag{84}$$

This choice of coordinates is called the *Lorentz gauge.* The Ricci tensor takes a very simple form under the Lorentz gauge.

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} \tag{85}$$

For vacuum solutions the Ricci scalar in our nearly-flat spacetime is zero, and the linearized Einstein equation takes the very simple form of a homogeneous wave equation:

$$G_{\mu\nu} = \Box h_{\mu\nu} = 0 \tag{86}$$

The solutions to this equation certainly include waves, and so we are (finally) where we wanted to be: General Relativity predicts the existence of free-space traveling waves. Just as Maxwell's equations predict the existence of oscillating electric and magnetic fields that travel through space, the Einstein equation predicts the existence of traveling ripples in spacetime itself. However, these are not oscillating fields *in space* like electromagnetic waves; in general relativity the field is the geometry, and the waves are traveling distortions in the spacetime manifold, in the curvature of that manifold. In a sense, the field *is space.* We have found this solution for the case of weak fields without sources present, but waves are also predicted when gravity is strong and sources are near. We will not explore that regime in any detail because, as mentioned earlier, it requires the use of numerical techniques. Instead we will look at some of the properties of the waves we have found. Only in the last section of this paper will we consider some generic properties that must be true of any source.

6.2 Properties of Gravitational Waves Under the Lorentz Gauge

We have found that under the conditions of the Lorentz gauge the Einstein equation takes the form of a wave equation:

$$\frac{\partial^2 h_{\mu\nu}}{\partial t^2} - \nabla^2 h_{\mu\nu} = 0 \tag{87}$$

This solution holds for small perturbations $h_{\mu\nu}$ on a background Minkowski (locally flat) metric. The solutions to this equation are of the form

$$h_{\mu\nu} = A_{\mu\nu}\exp(ik_\beta x^\beta) \tag{88}$$

For some (complex) amplitude $A_{\mu\nu}$ and wavenumber k_β, with $k_\beta = (\omega/c, k_x, k_y, k_z)$. We can write the wave equation in a slightly different form if we like:

$$\Box h_{\mu\nu} = h_{\mu\nu,\alpha}{}^{,\alpha} = h_{\mu\nu,\alpha\beta}\eta^{\beta\alpha} = 0 \tag{89}$$

Then, if we plug our solution (89) back into the wave equation (90) we find that

$$h_{\mu\nu,\alpha\beta}\eta^{\beta\alpha} = -k_\alpha k_\beta \eta^{\beta\alpha} h_{\mu\nu} = -k_\alpha k^\alpha h_{\mu\nu} = 0 \tag{90}$$

The only way for this to be true generally is if $k_\alpha k^\alpha = 0$, which means that k_α is a null-vector, or in other words, that its length is zero. So the **gravitational waves must travel at the speed of light**.

Homework Question 8

Use the definition of the wave-vector k^α to justify the last claim about the speed of gravitational waves.

We can also employ the Lorentz gauge condition in Equation 85 to impose several conditions on the waves and their amplitudes. For example, plugging our solution (Equation 89) into Equation 85 we find the following:

$$k_\beta A^\beta{}_\alpha - \frac{1}{2} A^\beta{}_\beta k_\alpha = 0 \tag{91}$$

Considering the case for $\alpha = 0$ we have

$$k_\beta A^\beta{}_0 - \frac{1}{2} A^\beta{}_\beta k_0 = 0 \tag{92}$$

We know that $k_0 \neq 0$ because k is a null vector. That means that the only way to satisfy Equation 93 generally is if $A^\mu{}_\mu = 0$, or in other words, **for A to be traceless**. This simplifies things immediately, since now we have

$$k_\beta A^\beta{}_\alpha = 0 \tag{93}$$

This is the dot product of the propagation direction (k) with the wave amplitude (A), and so from this we see immediately that the two are perpendicular; **these waves are transverse**, just like electromagnetic waves. And since $k_\beta A^\beta{}_0 = 0$, it must be that $A^\beta{}_0 = 0$ for all β, because $k_\beta \neq 0$. Additionally, we know from the symmetry of $h^\alpha{}_\beta$ that A must be symmetric too, thus we must have $A^0{}_\beta = 0$ for all β as well.

If we assume that our waves propagate in the $x^3\,(= z)$ direction, then we can use a similar analysis to show that $A^3{}_\beta = A^\beta{}_3 = 0$ for all β. So the amplitude A of the waves has a very simple form: $A_{12} = A_{21} \neq 0$ and $A_{11} = -A_{22} \neq 0$. All other components of A are zero. So we could write out A in matrix form as

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{21} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{94}$$

And if we wish to express the amplitude directly in terms of the perturbation, h, we have

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(h_{xx} - h_{yy}) & h_{xy} & 0 \\ 0 & h_{xy} & \frac{1}{2}(h_{yy} - h_{xx}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{95}$$

This solution for the perturbation is often called the *transverse-traceless gauge.*

Homework Question 9

Show that the matrix in Equation 96 has trace equal to zero, and explain why this gauge is called transverse.

6.3 Polarization

Looking at the last matrix in the previous section, you might suspect that gravitational waves can be decomposed into independent polarizations, just like electromagnetic waves. If you thought so, you were right. We can find the polarization components by considering how a passing gravitational wave will affect a freely falling test particle, or in other words, how it will affect a particle at rest in the background Minkowski space.

Considering solutions of Equation 80 in the Math Supplement in which h^{TT} refers to the matrix in Equation 96, we have two cases: (1) $h_\times = 0$ and (2) $h_+ = 0$. The values of α and β can only be 1 or 2.

Starting with case (1), $h_\times = 0$:

$$\frac{\partial^2 \xi_1}{\partial \tau^2} = \frac{1}{2}\xi_1 \frac{\partial^2}{\partial t^2}\left[h_+ \exp(ik_\alpha x^\alpha)\right] \tag{96}$$

$$\frac{\partial^2 \xi_2}{\partial \tau^2} = -\frac{1}{2}\xi_2 \frac{\partial^2}{\partial t^2}\left[h_+ \exp(ik_\alpha x^\alpha)\right] \tag{97}$$

Solutions for these equations are

$$\xi_1 = \xi_1(0)\left[1 + \frac{1}{2}h_+ \exp(ik_\alpha x^\alpha)\right] \tag{98}$$

$$\xi_2 = \xi_2(0)\left[1 - \frac{1}{2}h_+ \exp(ik_\alpha x^\alpha)\right] \tag{99}$$

For case (2) with $h_+ = 0$ we have similar equations to solve:

$$\frac{\partial^2 \xi_1}{\partial \tau^2} = \frac{1}{2}\xi_2 \frac{\partial^2}{\partial t^2}\left[h_\times \exp(ik_\alpha x^\alpha)\right] \tag{100}$$

$$\frac{\partial^2 \xi_2}{\partial \tau^2} = \frac{1}{2}\xi_1 \frac{\partial^2}{\partial t^2}\left[h_\times \exp(ik_\alpha x^\alpha)\right] \tag{101}$$

and the solutions are

$$\xi_1 = \left[\xi_1(0) + \frac{1}{2}\xi_2(0) h_\times \exp(ik_\alpha x^\alpha)\right] \tag{102}$$

$$\xi_2 = \left[\xi_2(0) - \frac{1}{2}\xi_1(0) h_\times \exp(ik_\alpha x^\alpha)\right] \tag{103}$$

From these solutions we see that gravitational waves have two possible polarizations. One, with amplitude h_+, has oscillations along the x and y axes, The other, $h_\times$, has oscillations rotated $45\,^\circ$ from the first, in striking contrast to electromagnetic waves which have linear polarization planes separated by $90\,^\circ$. The two orientations are shown in Figure 7.

6.4 Sources of Gravitational Waves

We now look at some of the basic properties of the sources that we expect to generate gravitational waves. We will not concern ourselves with details; those are the subject of the next part of the course. Instead, we will look only at the general nature of the sources and how sources of gravitational waves differ from sources of electromagnetic waves.

In electromagnetic theory, we find that the strongest sources of waves come from the time derivatives of the dipole field terms. These can be either the electric dipole $\mathbf{d}$ or magnetic dipole $\boldsymbol{\mu}$, which are defined as

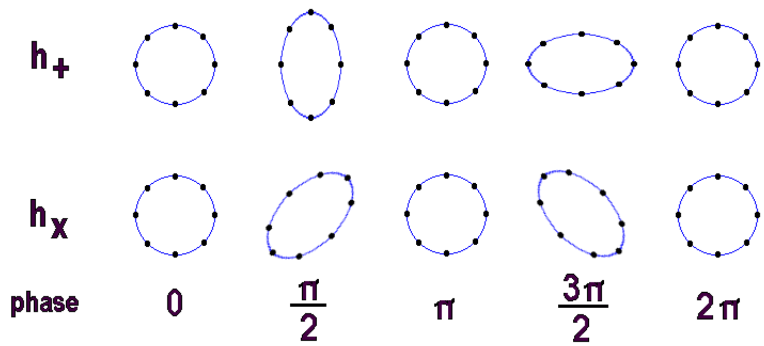

Figure 7: This shows the h_+ and $h_\times$ polarizations for gravitational waves. Notice how the waves alternately stretch and then compress the spacetime. Image: http://www.johnstonsarchive.net/relativity/pictures.html.

$$\mathbf{d} = \sum_a e_a \mathbf{r}_a \tag{104}$$

$$\boldsymbol{\mu} = \sum_a e_a \mathbf{r}_a \times \mathbf{v}_a \tag{105}$$

We assume the a^{th} particle has electric charge e_a, sits at position $\boldsymbol{r}_a$ and has velocity $\mathbf{v}_a$. For continuous charge distributions we replace the sum with an integral over charge density. We can explore how these terms might contribute to gravitational radiation if we replace the charge of the particles, e_a, with their masses, m_a. What we will find is that the dipole terms of the fields make no contribution to the production of gravitational waves, as suggested in HW Question 10.

Homework Question 10

Rewrite equations 105 and 106 to use masses, m_a instead of charge e_a. Compute their time derivatives. Why do these derivatives vanish for the case of gravity, while they do not vanish for electromagnetism?

The next strongest term in the multipole expansion of the field is the quadrupole, so we can check to see if that provides a non-vanishing contribution to the radiation. The definition of the quadrupole moment is

$$I_{ij} = \sum_a m_a x_{ai} x_{aj} \tag{106}$$

In order to understand how this might play into the production of gravitational waves we must solve the Einstein equation with sources present. We will continue to limit ourselves to waves in a region of nearly flat space. This means that the source of the waves must be at a great distance from the point at which we detect them, or in other words, that we are many, many wavelengths away from the source. Further, we will assume that the waves measured are dominated by mass, not by the gravitational field produced by the mass or by the kinetic energy of the source. These conditions imply that the wavelengths of the waves we study are much longer than the size of the source. With these assumptions, the Einstein equation becomes

$$\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \tag{107}$$

The bar over the h indicates that we must take the trace reverse of h:

$$\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \tag{108}$$

The solution to this equation has the generic form

$$\bar{h}^{\mu\nu} = \frac{4\pi G}{c^4} \int \frac{T^{\mu\nu}(\mathbf{x}', t^{ret})}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{109}$$

where the integral is taken over all space, but its effects at the position $\mathbf{x}$ are only taken at the retarded time, $t^{ret} \equiv t - |\mathbf{x} - \mathbf{x}'|/\mathbf{c}$; in this way general relativity differs from Newtonian gravity because it does not allow instantaneous changes to the field in all space when the source, $T^{\mu\nu}(\mathbf{x}', t)$, undergoes some change.

We must evaluate this integral in order to find out what the value of $\bar{h}^{\mu\nu}$ will be. Under the assumptions outlined above, we are at a distance r much greater than the size of our source, so we can write

$$\bar{h}^{\mu\nu} = \frac{4\pi G}{rc^4} \int T^{\mu\nu}(\mathbf{x}', t^{ret}) d^3x' \tag{110}$$

We can employ the conservation of mass-energy to rewrite the right hand side of this equation as follows. We know that

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{111}$$

and from this we have

$$T^{00}{}_{,0} = -T^{0l}{}_{,l} \tag{112}$$

Differentiating this with respect to time, and using the symmetry of $T^{\mu\nu}$ and the commutativity of partial derivatives we get

$$T^{00}{}_{,00} = -T^{0l}{}_{,l0} = -T^{l0}{}_{,0l} \tag{113}$$

Now we can again use the conservation of mass-energy to obtain

$$T^{m0}{}_{,0m} = \left(T^{m0}{}_{,0}\right)_{,m} = -T^{ml}{}_{,lm} \tag{114}$$

So substituting this into Equation 114 we have

$$(T^{00})_{,00} = T^{ml}{}_{,lm} \tag{115}$$

If we multiply Equation 116 by $x^k x^j$ we find the following:

$$(T^{00} x^j x^k)_{,00} = T^{ml}{}_{,lm} x^j x^k \tag{116}$$

We are able to take x^j and x^k inside the derivative on the left because they do not depend on the time. We wish to rewrite the right-hand side of this equation, and we can do so if we multiply T^{ml} by x^j and x^k and then differentiate with respect to x^l and x^m. We will then find the identity[3]

$$\partial_l \partial_m \left(T^{lm} x^j x^k\right) = x^j x^k \partial_l \partial_m T^{lm} + 2\partial_l \left(x^k T^{jl} + x^j T^{kl}\right) - 2T^{jk} \tag{117}$$

Notice that the first term on the right-hand side is the same as the right-hand side of Equation 117. Using this identity to replace that term we get

$$T^{00} x^j x^k{}_{,00} = \partial_l \partial_m \left(T^{lm} x^j x^k\right) - 2\partial_l \left(x^k T^{jl} - x^j T^{kl}\right) + 2T^{jk} \tag{118}$$

We may now integrate this equation over all space. When we do so, the first two terms on the right-hand side vanish. The first can be converted to a surface integral using the divergence theorem. Evaluating at large r where $T^{lm} = 0$ causes it to vanish. The second term vanishes due to its odd symmetry. So the only term left is the last one, containing T^{jk}. Thus, after integrating we are left with

$$2 \int T^{jk} d^3x = \int \frac{\partial^2}{\partial t^2} T^{00} x^j x^k d^3x = \frac{\partial^2}{\partial t^2} \int T^{00} x^j x^k d^3x \tag{119}$$

The term inside the integral on the far right is called the quadrupole moment tensor, Q^{jk}, of the mass distribution of the source. So we find that the equation becomes

$$\frac{1}{2} \ddot{Q}^{jk} = \int T^{jk} d^3x \tag{120}$$

[3]Thanks to J. Creighton of the University of Wisconsin, Milwaukee for helping track down a pesky sign error in this identity, now fixed.

The dots above **Q** indicate time differentiation.

Finally, substituting this expression into Equation 111 we get

$$\bar{h}^{jk} = \frac{2\pi G}{rc^4}\ddot{Q}_{jk} \tag{121}$$

So the lowest-order multipole contributing to gravitational waves is the quadrupole, and only if the source has a time-varying quadrupole moment do we expect it to emit detectable gravitational waves.

7 Conclusion

This concludes our look at the theoretical aspects of gravitational waves. Our purpose was to give an overview of how gravitational waves are produced as a natural solution of the Einstein equation, just as electromagnetic waves are a natural prediction of Maxwell's equations. Much of this section has been involved with developing, in a very limited way, the basic mathematical concepts needed to understand the Einstein equation and its solutions. In particular, unlike classical electromagnetism, the Theory of General Relativity is not concerned with fields and their properties *in space*. Rather, general relativity can be thought of as a non-linear field theory concerning the curvature of spacetime itself, and it is spacetime that *is the field*. This is a fundamentally different perspective, and the mathematical tools needed to deal with curvature of spacetime are quite a bit different than those learned in other branches of physics. We have only given the briefest introduction to these tools and the ideas behind them. However, we hope that we have peaked your curiosity to the point that you will want to work through some of the very good books on general relativity (see below) that provide a more complete introduction than we have been able to provide here.

The remainder of the course will be concerned with the kinds of astronomical sources we expect to observe in the universe.

7.1 Further Reading

Several books were used as references while preparing this paper. They are listed below. The first two have excellent discussions of curvature, with the first placing more emphasis on the mathematics from the outset, and the second making better connections to the physics as it proceeds through the material. Both are written at the college senior / first year graduate level. The third book was also used, mostly because the author, Carroll, has excellent mathematical and physical insights to share. The mathematical level is higher in Carroll than in the other two books, and it seems more for a graduate student who will be going into GR, or who will have reason to be well acquainted with the ideas and methods of GR. Any of these three books would be a good investment for a person interested in developing their understanding of the General Theory of Relativity. The fourth book is the bible of gravitation. It is not a good book to begin learning about gravity, at least I have not found it so. It is, however, an excellent one for refining one's ideas once some basic understanding has been gained. MTW contains essentially anything you might like to know about general relativity. Use it as you might use the Feynman Lectures to increase your physical insights, and also to learn more of the detailed mathematics if that is your interest.

Schutz, Bernard F., *A First Course in General Relativity, 2nd Edition*, Cambridge University Press, 2009

Hartle, James B., *Gravity: An Introduction to Einstein's General Relativity*, Addison Wesley, 2003

Moore, Thomas A., *A General Relativity Workbook*, University Science Books, 2013

Carroll, Sean., *Spacetime and Geometry: An Introduction to General Relativity*, Addison Wesley, 2004

Misner, C. W. , Thorne, K. S, and Wheeler, J. A, *Gravitation*, Freeman, 1973

This last reference is a nice overview, with good explanations of some of the concepts and mathematical tools that are important in GR. It is linked in the **Additional Resources: General Relativity** box on the course Moodle site.

Price, Richard H., *General Relativity Primer*, American Journal of Physics, **50**, 300, 1982

In addition, the websites of Marc Favata, of the LIGO science team, has lots of good references on general relativity and gravitational waves. Professor Favata gave us useful feedback on the content of this document and improved it greatly by doing so. Nonetheless, any errors are entirely the author's.

Gravitational Wave Resources
`http://www.astro.cornell.edu/~favata/gwresources.html`

Gravitational Wave Outreach Resources
`http://www.astro.cornell.edu/~favata/outreach.html`

For books dealing specifically with special relativity and flat spacetime, the following two are excellent.

Taylor, E. F. and Wheeler, J. A, *Spacetime Physics: Introduction to Special Relativity, 2nd Ed*, Freeman, 1992

Moore, T. A, *Six Ideas that Shaped Physics, Unit R: The Laws of Physics Are Frame-Independent, 2nd Ed.*, McGraw-Hill, 2003

References

Marolf, D. 2003, Relativity and cosmology for phy312. Course notes

Schutz, B. F. 2009, A First Course in General Relativity (Cambridge: Cambridge University Press), 2nd ed.