Learning More About Light

July 2, 2015

1 Maxwell's Equations Overview

Recall the differential forms of Maxwell's equations as they are usually taught in college physics:

$$\nabla \cdot \overrightarrow{E} \; = \; \frac{\rho}{\epsilon_o} \tag{1}$$

$$\nabla \times \overrightarrow{E} \; = \; - \frac{\partial \overrightarrow{B}}{\partial t} \tag{2}$$

$$\nabla \times \overrightarrow{B} \; = \; \mu_o \epsilon_o \frac{\partial \overrightarrow{E}}{\partial t} \; + \; \mu_o \overrightarrow{J} \tag{3}$$

$$\nabla \cdot \overrightarrow{B} \; = \; 0 \tag{4}$$

and also the Lorentz force equation:

$$\overrightarrow{F} = q(\overrightarrow{E} \; + \; \overrightarrow{v} \times \overrightarrow{B}) \tag{5}$$

In these equations, $\overrightarrow{E}$ is the electric field (vector), and $\overrightarrow{B}$ is the magnetic field (vector). Together with the Lorentz force equation, Maxwell's equations contain all our knowledge of classical electricity and magnetism. Interpreting them, we find that electric field lines start and end on charges (ρ), changing magnetic fields will create electric fields, changing electric fields and currents ($\overrightarrow{J}$) will create magnetic fields, and that magnetic monopoles do not exist.

Most problems in calculus-based college physics use laws that can be derived from Maxwell's equations, such as Gauss' Law, Faraday's Law, Ampere's Law, and the Biot-Savart Law. Almost all of these problems include various types of charge and current distributions, and integration over surfaces to determine the field geometry. In this class, we are not concerned with the static charge distributions that are the mainstay in typical college classes. Rather, we wish to understand the electromagnetic field in regions free of sources.

2 Maxwell's Equations in Source-Free Space

Without current and charge, the differential forms of Maxwell's Equations become:

$$\nabla \cdot \overrightarrow{E} = 0 \tag{6}$$

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{7}$$

$$\nabla \times \overrightarrow{B} = \mu_o \epsilon_o \frac{\partial \overrightarrow{E}}{\partial t} \tag{8}$$

$$\nabla \cdot \overrightarrow{B} = 0 \tag{9}$$

Consider the coupling between the two equations with both $\overrightarrow{E}$ and $\overrightarrow{B}$ and let's try to eliminate $\overrightarrow{B}$ by taking the curl of both sides of the second equation above:

$$\nabla \times (\nabla \times \overrightarrow{E}) = -\frac{\partial}{\partial t}(\nabla \times \overrightarrow{B}) \tag{10}$$

In the above equation, the order of differentiation with respect to space and time has been interchanged on the right hand side. We can now insert Equation 8 above into this equation to eliminate the magnetic field:

$$\nabla \times (\nabla \times \overrightarrow{E}) \;=\; -\;\mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2} \tag{11}$$

In order to evaluate this equation, we need to recall the identity;

$$\nabla \times (\nabla \times \overrightarrow{E}) \;=\; \nabla(\nabla \cdot \overrightarrow{E}) - (\nabla \cdot \nabla)\overrightarrow{E} \tag{12}$$

But in source-free space, we know from Equation 6 above, that $\nabla \cdot \overrightarrow{E} = 0$, so we have

$$\nabla^2\overrightarrow{E} = \mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2} \tag{13}$$

In Cartesian coordinates, this can also be written:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2} \tag{14}$$

This is the wave equation for the electric field, and is really three equations (*i.e.*, one for each Cartesian dimension).

Homework 1

Derive the wave equation for $\overrightarrow{B}$ in a similar manner. It may be helpful to refer to the background handout about vector operators. Feel free to do your work with pencil and paper, then take a photo and upload it or scan it in and submit it using the Moodle.

3 Electromagnetic Waves

The two vector wave equations for electric and magnetic fields, when considered together have a very powerful implication: electromagetic waves exist and do not require any physical matter through which to propagate. They can exist in a perfect vacuum. We call them electromagnetic waves because each wave is characterized by two interdependent oscillating quantities: the electric field and the magnetic field.

Compare Equation 13 to the standard wave equation (that also describes mechanical waves, such as waves on strings, springs, etc.):

$$\nabla^2\psi \ = \ \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \tag{15}$$

We can immediately see that the velocity of propagation v of our electromagnetic wave is equal to:

$$v \ = \ \frac{1}{\sqrt{\mu_o\epsilon_o}} \tag{16}$$

And if we use the experimentally measured values for $\mu_o \ = \ 4\pi \times 10^{-7}$ webers m^{-1} amp^{-2} and $\epsilon_o \ = \ 8.854 \times 10^{-12}$ coulombs volt^{-1} m^{-2}, we find that $v \ = \ 2.9980 \times 10^8$ meter sec^{-1}, which is the speed of light. Maxwell realized this in 1865 and we now know that these equations apply to the entire electromagnetic spectrum, from radio waves through gamma rays.

Homework 2

How do the oscillation directions of the electric and magnetic field components compare to each other? Explain (qualitatively). You do not need to show the answer mathematically (that is *Homework Question 4*).

For a plane sinuosoidal wave traveling along +z, one solution to the wave equation is given by:

$$E(x,t) \ = \ E_{ox}\cos(\omega t \ - \ kz + \phi) \tag{17}$$

AND

$$B(y,t) \ = \ \frac{E_{ox}}{c}\cos(\omega t \ - \ kz + \phi) \tag{18}$$

where ω is the angular frequency, $k = \omega/c$ is the wave number and ϕ is an arbitrary phase.

Homework 3

Show that Equation 17 satisfies the wave equation for electromagnetic waves given in Equation 14.

A generalized solution to the vector wave equation given in Equation 13 is:

$$\overrightarrow{E}(\overrightarrow{r},t) = \overrightarrow{E_o}e^{i(\omega t-\overrightarrow{k}\cdot\overrightarrow{r})} \tag{19}$$

where $\overrightarrow{k}$ indicates the propagation direction, and the wave is assumed to have an amplitude that is not a function of the time or spatial coordinates. The corresponding solution for the magnetic field is:

$$\overrightarrow{B}(\overrightarrow{r},t) = \overrightarrow{B_o}e^{i(\omega t-\overrightarrow{k}\cdot\overrightarrow{r})} \tag{20}$$

Homework 4

Show that (in empty space) $\overrightarrow{E_o}$, $\overrightarrow{B_o}$ and $\overrightarrow{k}$ (as defined in Equations 19 and 20) are mutually orthogonal and obtain the relation between $\overrightarrow{E_o}$ and $\overrightarrow{B_o}$.

4 Energy in Electromagnetic Waves

Electromagnetic waves are *transverse* - that is the directions of the oscillating electric and magnetic fields are orthogonal to each other and to the direction of propagation of the wave. We define the propagation direction by the Poynting vector:

$$\overrightarrow{S} = \frac{1}{\mu_o}(\overrightarrow{E}\times\overrightarrow{B}) \text{ watt m}^{-2} \tag{21}$$

The direction of the Poynting vector is given by the right-hand rule, and also points out the direction in which energy flows. For our previous example, with the electric field oscillating in the x-direction, and the magnetic field oscillating in the y-direction, the magnitude of the instantaneous Poynting flux will be given by:

$$S_z = \epsilon_o c E_{ox}^2 cos^2(\omega t \ - \ kz + \phi) \tag{22}$$

and will vary periodically between 0 and E_{ox}^2 with a frequency of 2ω.

5 Sources of Electromagnetic Radiation

Far from a charge or system of charges, we expect to see a spherical wave propagating outward and transporting energy. In a vacuum (like space), the total energy carried through a sphere of radius r should be independent of the size of r since the wave cannot gain or lose energy to the vacuum. Since the area of a sphere with radius r is $4\pi r^2$, the time averaged Poynting flux must vary like $1/r^2$ so that the product of $< |\overrightarrow{S}| >$ with the area will be invariant and energy will flow through space. Since we have shown previously that $\overrightarrow{E}$, $\overrightarrow{B}$ and $\overrightarrow{S}$ are all orthogonal, and that the magnitude of $\overrightarrow{B}$ is equal to $\frac{\overrightarrow{E}}{c}$, we require a sinusoidal spherical electromagnetic wave that takes the following form as $r \rightarrow \infty$:

$$|\overrightarrow{E}| \ = \ \frac{\text{“strength of source”}}{\text{r}} \cos(\omega t - kr) \tag{23}$$

$$|\overrightarrow{B}| \ = \ \frac{\text{“strength of source”}}{\text{rc}} \cos(\omega t - kr) \tag{24}$$

The following derivation for the “strength of source” term is taken from Bekefi and Barrett (1977). This term includes all the physics by which an accelerating charge generates an electromagnetic wave. It is important to realize that a stationary charge cannot generate an electromagnetic wave: its field lines are radial, and there is no corresponding magnetic field. Therefore the Poynting flux is zero in this case.

Similarly, a charge moving at a constant velocity also cannot radiate, even though moving charges create currents, and therefore magnetic fields. However, at a constant velocity, the electric field still points radially out from the charge, and both the electric and magnetic field strengths fall off like $\frac{1}{r^2}$, which will create a Poynting flux that falls off like $\frac{1}{r^4}$. The energy flowing through a sphere with radius r will therefore fall off like $\frac{1}{r^2}$ and will not remain constant as $r \rightarrow \infty$. Figure 1 shows the radiation pattern from a positive charge moving at constant velocity. In case (a), the speed is low while in case (b) it is relativistic. In both cases, the field lines still point radially.

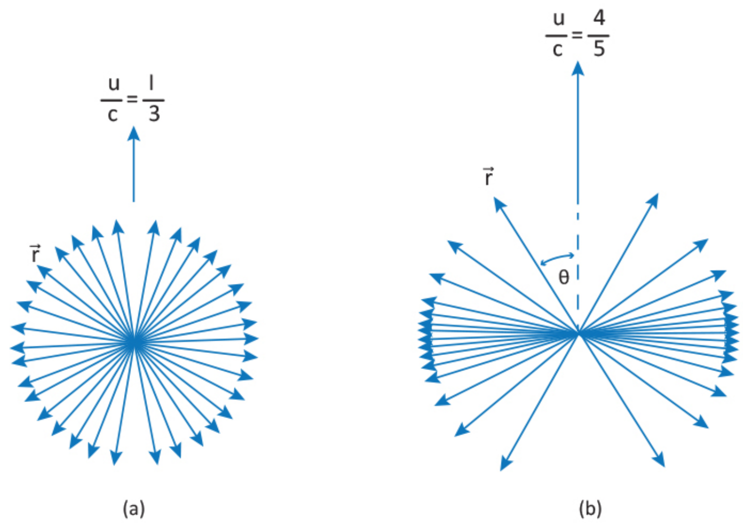

Figure 1: Electric field lines for positively charged particle moving at a constant velocity.[1]

Consider a point charge $+q$ at rest at the origin. Suddenly, at time $t = 0$, the charge is given an acceleration a for a very short time Δt. It reaches point 1. Then it coasts at the new velocity $u = a\Delta t$ and continues to move at this speed along the x-axis for a time t until it reaches point 2. What is the field at the end of time t when the charge is at point 2? The situation is summarized in Figure 2.

In Figure 2, the line between the origin and point A is the field line from the charge $+q$ while it was stationary at 0. We draw a sphere of radius $c(t + \Delta t$ around the origin. The field emitted at the beginning of the acceleration period Δt has just had time to reach the surface of this sphere. An observer located outside this sphere will therefore not yet have realized that the charge was accelerated.

Now draw a second sphere of radius ct with its center at point 2. At point 2, the field line to this sphere is again radially outward, intersecting the sphere at a radial distance that is shorter by $c\Delta t$. However, the field lines cannot stop or start in empty space, and so the original field line (emanating from point 0) must connect to the field line (emanating at point 2) within the shell. Joining these two field lines creates a "kink". As time goes on, the shell with the kink propagates outward at the speed of light. It is this kink which

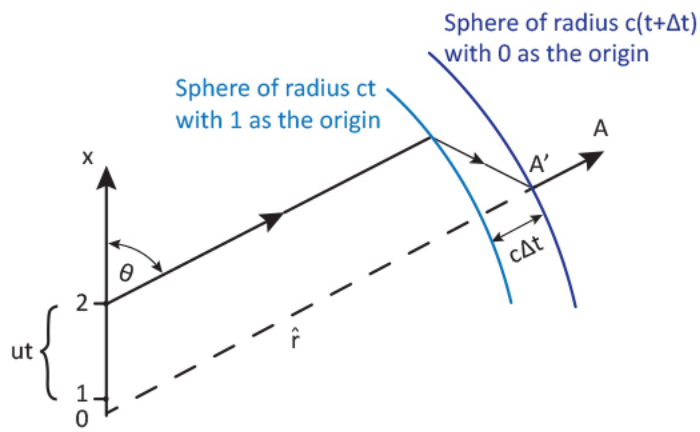

Figure 2: Electric field lines for positively charged particle accelerating for a short time.[1]

carries energy into space. Note that we have assumed that the velocity u is very small compared to the speed of light c, so that the two radial field lines are virtually parallel to each other and to the radius vector $\overrightarrow{r}$.

If we next examine the components of the electric field, we see that it can be resolved into components that are parallel and perpendicular to the radius vector. Using similar triangles as shown in Figure 3, we see that:

$$\frac{E_\perp}{E_\parallel} = \frac{u_\perp t}{c\Delta t} \tag{25}$$

where $u_\perp$ is the component of the charge's velocity that is perpendicular to $\overrightarrow{r}$. Since $u = a\Delta t$, it follows that $u_\perp = a_\perp \Delta t$ where $a_\perp$ is the component of the acceleration perpendicular to $\overrightarrow{r}$.

Homework 5

Use Equation 25, and $r = ct$ to show that:

$$E_\perp = E_\parallel \left(\frac{a_\perp r}{c^2}\right) \tag{26}$$

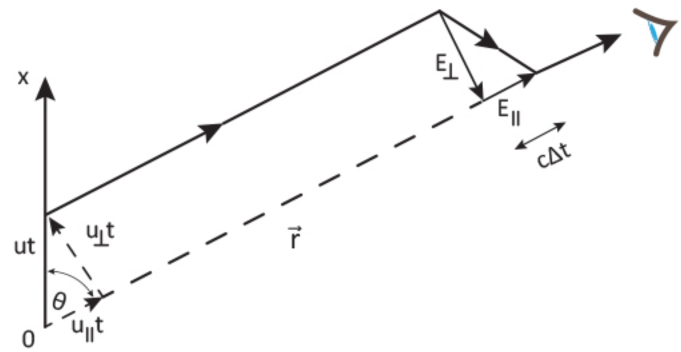

Figure 3: The electric field resolved into components parallel and perpendicular to the radius vector.[1]

Since there is no charge contained in the Gaussian pillbox shown in Figure 4 below, the magnitude of $E_{\|}$ is given by the strength of the original radial field:

$$E_{\|} \;=\; E_o \;=\; \frac{q}{4\pi\epsilon_o r^2} \tag{27}$$

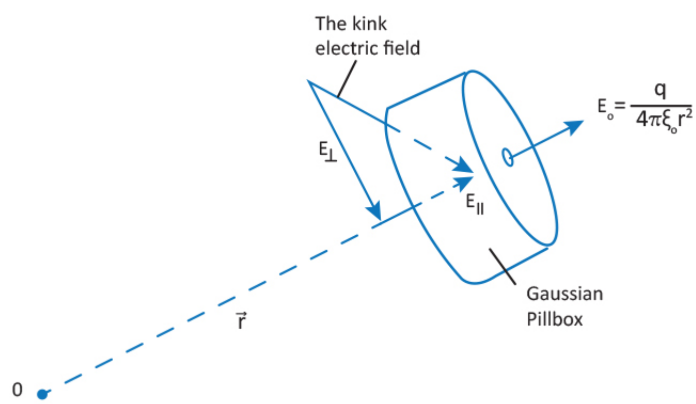

Figure 4: Gaussian pillbox geometry.[1]

Homework 6

Use Equation 27 together with Equation 26 to show that the final result for the electric field from the accelerating charge is transverse to the propagation direction $\overrightarrow{r}$, and that it falls off like $\frac{1}{r}$. What term replaces the words "strength of source" in Equation 23?

6 Dipole Approximation

In the derivation above, we have used a single point charge. However, this derivation is also valid for an extended charge distribution, as long as the size of the distribution is much smaller than the wavelength of the radiation $\lambda = 2\pi c/\omega$, and as long as relativistic corrections on the order of $\frac{u}{c^2}$ can be neglected. It must also hold that the observation distance r must be much greater than the wavelength of the radiation λ. When all these conditions hold, the vector equations for the electric field, magnetic field and Poynting flux are given by:

$$\overrightarrow{E}(\overrightarrow{r},t) = -\frac{q\overrightarrow{a}_{\perp}(t')}{4\pi\epsilon_o r c^2} \ \text{volt m}^{-1} \tag{28}$$

$$\overrightarrow{B}(\overrightarrow{r},t) = \hat{r} \times \overrightarrow{E}(\overrightarrow{r},t)/c \ \text{webers m}^{-2} \tag{29}$$

$$\overrightarrow{S} = \frac{\overrightarrow{E} \times \overrightarrow{B}}{\mu_o} \ \text{watts m}^{-2} \tag{30}$$

This is known as the dipole approximation to the radiation field. Note that $\overrightarrow{E}(\overrightarrow{r},t)$ is due to the acceleration $\overrightarrow{a}_{\perp}(t')$ which took place at an earlier time $t' = t - r/c$. In other words, the change in the electric field geometry does not propagate instantly to a distant observer. The "kink" in the field travels at the speed of light.

Let θ be the angle between the instantaneous direction of the acceleration vector $\overrightarrow{a}$ and the direction of the wave propagation $\overrightarrow{r}$. In this case, the perpendicular component of the acceleration $a_{\perp}$ is just $a \sin\theta$ and

$$E(r,t) = -\frac{qa(t')\sin\theta}{4\pi\epsilon_o r c^2} \ \text{volt m}^{-1} \tag{31}$$

Homework 7

Sketch the radiation pattern given by Equation 31 and verify that it has the expected dipole shape, where the maximum radiation is seen perpendicular to the direction of acceleration.

References

[1] Bekefi, George and Barrett, Alan H., *Electromagnetic Vibrations, Waves and Radiation*, MIT Press, Cambridge, MA (1977).