

Vector Calculus Operators

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1 Div, Grad and Curl

The divergence of a vector quantity in Cartesian coordinates is defined by:

$$\nabla \cdot \vec{a} \equiv \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \quad (1)$$

where $\vec{a} = (a_x, a_y, a_z)$. The divergence of a vector quantity is a scalar quantity. If the divergence is greater than zero, it implies a net flux out of a volume element. If the divergence is less than zero, it implies a net flux into a volume element. If the divergence is exactly equal to zero, then there is no net flux into or out of the volume element, *i.e.*, any field lines that enter the volume element also exit the volume element. The divergence of the electric field and the divergence of the magnetic field are two of Maxwell's equations.

The gradient of a scalar quantity in Cartesian coordinates is defined by:

$$\nabla \phi(x, y, z) \equiv \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \quad (2)$$

where \hat{i} is the unit vector in the x-direction, \hat{j} is the unit vector in the y-direction, and \hat{k} is the unit vector in the z-direction.

Evaluating the gradient indicates the direction in which the function is changing the most rapidly (*i.e.*, slope). If the gradient of a function is zero, the function is not changing.

The curl of a vector function in Cartesian coordinates is given by:

$$\nabla \times \vec{f} \equiv \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \hat{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) \hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \hat{k} \quad (3)$$

where the resulting function is another vector function that indicates the circulation of the original function, with unit vectors pointing orthogonal to the plane of the circulation components. The curl of the electric field and the curl of the magnetic field are two of Maxwell's equations.

The Laplacian of a scalar function in Cartesian coordinates is given by:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (4)$$

The Laplacian operator is important in many physics equations, including the wave equation.

2 Vector Operator Identities

The vector triple product is defined by:

$$\vec{A} \times (\vec{B} \times \vec{C}) \equiv \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \quad (5)$$

If we substitute the ∇ operator in for the first two vectors in this identity, we can apply the vector triple product to the electric field as follows:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla) \vec{E} \quad (6)$$

The scalar triple product can be written as $[\vec{A}; \vec{B}; \vec{C}]$ and can take three equivalent forms:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad (7)$$

One easy way to evaluate the scalar triple product is by taking the determinant of the 3×3 matrix $[\vec{A}; \vec{B}; \vec{C}]$:

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The determinant of this matrix, $\det(\vec{A} \ \vec{B} \ \vec{C})$, is equal to:

$$A_x B_y C_z + A_y B_z C_x + A_z B_x C_y - A_z B_y C_x - A_x B_z C_y - A_y B_x C_z \quad (8)$$

3 Operators in Four Dimensional Spacetime

In four-dimensional spacetime, the combination of the three-dimensional Laplacian with the second-time derivative term is called the d'Alembertian, which is defined as:

$$\square^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad (9)$$

where c is the speed of light.

References

- [1] Mathworld.wolfram.com